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EFFECT OF THE THICKNESS VARIATION AND INITIAL IMPERFECTION ON BUCKLING OF COMPOSITE CYLINDRICAL SHELLS: ASYMPTOTIC ANALYSIS AND NUMERICAL RESULTS BY BOSOR4 AND PANDA2

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Abstract—This study is an extension of a previous investigation of the combined effect of axisymmetric thickness variation and axisymmetric initial geometric imperfection on buckling of isotropic shells under uniform axial compression. Here the anisotropic cylindrical shells are investigated by means of Koiter's energy criterion. An asymptotic formula is derived which can be used to determine the critical buckling load for composite shells with combined initial geometric imperfection and thickness variation. Results are compared with those obtained by the software packages BOSOR4 and PANDA2. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Due to various factors in the manufacturing process, thin cylindrical shells may exhibit variations in wall thickness. In spite of the fact that buckling of uniformly compressed cylindrical shells has been studied intensively for the past several decades, the influence of thickness variation on the buckling load has seldom been studied. In the previous research, we have investigated the effect of thickness variation on the axial buckling of otherwise perfect isotropic shells (Koiter *et al.*, 1994a) and imperfect isotropic shells (Koiter *et al.*, 1994b). These studies resulted in a conclusion that, although the thickness variation pattern in the form of the classical axisymmetric buckling mode may have some deleterious effect on the load-bearing capacity, the most detrimental effect of thickness variation occurs when the wave number of the axisymmetric thickness variation pattern is *twice* that of the classical buckling mode. Asymptotic relationships between the buckling load reduction rate and the thickness variation parameter were established for isotropic shells of non-uniform thickness (Koiter *et al.*, 1994a, 1994b).

The present study aims at the *combined* effect of axisymmetric thickness variation and axisymmetric initial imperfection on the buckling behavior of *composite* shells. We approach this problem by using Koiter's energy criterion of elastic stability (Koiter, 1945, 1966, 1980). Here, we consider the small axisymmetric thickness variation, and as a first approximation, only terms up to the first order of thickness variation parameter are retained. The final product of this discussion is again an asymptotic formula which relates the thickness variation parameter and initial imperfection amplitude to the buckling load of the structure. Therefore, this study is a direct generalization and extension of our former investigation (Koiter *et al.*, 1994a, 1994b) to the anisotropic case. The asymptotic formula obtained herein encompasses the isotropic shell. Comparisons with results obtained by the computer codes BOSOR4 and PANDA2 are provided.

2. FORMULATION BY THE ENERGY CRITERION

The nonlinear strain-displacement relations for cylindrical shells are

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \kappa_x = -\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \qquad \kappa_y = -\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \qquad \kappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} \tag{1}$$

where x and y are the axial and circumferential coordinates in the shell middle surface; u and v are the shell displacements along axial and circumferential directions, and w is the radial displacement, positive outward; ε_x, ε_y and γ_{xy} are strain components; κ_x, κ_y and κ_{xy} are middle surface curvatures of the shell; R is the radius of the cylinder.

Thickness variation of the laminated shell invariably exists due to imprecision involved in the fabrication process. Here we discuss the case that the thickness variation is axisymmetric and of uniform configurational nature: each lamina has the same variational pattern:

$$h_k(x) = h_{0,k}\left(1-\varepsilon\cos\frac{2p_1 x}{R}\right) = h_{0,k}H(x) \quad (k = 1 \sim K) \tag{2}$$

where h_k and $h_{0,k}$ are the thickness and the nominal thickness for the k-th layer, respectively; ε and p_1 are the non-dimensional parameters indicating the magnitude and wave number of the thickness variation, assumed to be the same for all the constituent layers; K represents the total number of layers in the laminate. At first sight, the perfect homology of the thickness variation may appear as a restrictive assumption. If the constituent layers are produced by the same manufacturing process according to the same specification, one can not rule out the existence of similar deviations from uniform thickness. Such an assumption may shed some light to the question of thickness variation and lead to a tractable analysis. However, most shells are manufactured by being wound on a mandrel. The inner wall of the shell would probably be flat and the outer wall would have all the thickness variation. In future, some numerical results will be reported for a variable thickness case where the inner surface is at a constant radius and all the thickness variation occurs on the outer surface. Here we assume that the middle surface of the shell with thickness variation only (no geometric imperfection) forms a perfect cylinder.

With the model presented in eqn (2), elements of the stiffness matrices $[A]$, $[B]$ and $[D]$ for the laminated shell with variable thickness become

$$A_{ij} = \sum_{k=1}^{K}(\bar{Q}_{ij})_k(h_k - h_{k-1}) = H(x)\sum_{k=1}^{K}(\bar{Q}_{ij})_k(h_{0,k} - h_{0,k-1}) = H(x)a_{ij}$$

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{K}(\bar{Q}_{ij})_k(h_k^2 - h_{k-1}^2) = \frac{1}{2}[H(x)]^2\sum_{k=1}^{K}(\bar{Q}_{ij})_k(h_{0,k}^2 - h_{0,k-1}^2) = [H(x)]^2 b_{ij}$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{K}(\bar{Q}_{ij})_k(k_k^3 - h_{k-1}^3) = \frac{1}{3}[H(x)]^3\sum_{k=1}^{K}(\bar{Q}_{ij})_k(h_{0,k}^3 - h_{0,k-1}^3) = [H(x)]^3 d_{ij}$$

$$(i,j = 1,2,6) \tag{3}$$

where a_{ij}, b_{ij} and d_{ij} are elements of stiffness matrices for the corresponding uniform laminate with thickness h_0; $\bar{Q}_{ij}$'s are the transformed reduced stiffnesses of the individual lamina and do not depend on the thickness. In the following, use will be made of the transformed stiffness matrices $[A^*]$, $[B^*]$ and $[D^*]$, which are related to the matrices in (3) as follows:

$$[A^*] = [A]^{-1}, \ [B^*] = [B][A], \ [D^*] = [D]-[B][A^*][B] \tag{4}$$

Thus

$$A^*_{ij} = \frac{1}{H(x)} a^*_{ij}, \quad B^*_{ij} = H(x) b^*_{ij} D^*_{ij} = [H(x)]^3 d^*_{ij} \tag{5}$$

where $[a^*]$, $[b^*]$ and $[d^*]$ are counterparts, in the uniform laminate, of the transformed stiffness matrices $[A^*]$, $[B^*]$ and $[D^*]$. They are given by

$$[a^*] = [a]^{-1}, \ [b^*] = [b][a], \quad [d^*] = [d]-[b][a^*][b]. \tag{6}$$

We will deal with symmetric laminates, for which there is no coupling between bending and extension. Thus, we have

$$B_{ij} = 0, \quad B^*_{ij} = 0 \quad (i,j = 1,2,6). \tag{7}$$

The constitutive relations for the anisotropic laminate are (Vinson and Sierakowski, 1986)

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{pmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} \tag{8}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{pmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{pmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{9}$$

where N_x, N_y and N_{xy} are stress resultants, M_x, M_y and M_{xy} are bending and twisting moments, acting on the mid-surface of a laminate.

Membrane strain energy of a laminated cylindrical shell of length L is

$$U_m = \frac{1}{2}\int_0^{2\pi R}\int_0^L (N_x\varepsilon_x + N_y\varepsilon_y + N_{xy}\gamma_{xy})\mathrm{d}x\mathrm{d}y. \tag{10}$$

Bending strain energy reads

$$U_b = \frac{1}{2}\int_0^{2\pi R}\int_0^L (M_x\kappa_x + M_y\kappa_y + M_{xy}\kappa_{xy})\mathrm{d}x\mathrm{d}y. \tag{11}$$

For the shell under axial uniform end compression N_0, potential energy of the applied load takes the form

$$\Omega = -\frac{1}{2}\int_0^{2\pi R}\int_0^L N_0\left(\frac{\partial w}{\partial x} + \frac{\partial w_0}{\partial x}\right)^2 \mathrm{d}x\mathrm{d}y \tag{12}$$

where w_0 is the geometric initial imperfection.

Thus, the total potential energy is

$$\Pi = U_m + U_b + \Omega \tag{13}$$

or, with use of the constitutive relations (8) and (9):

$$\Pi = \frac{1}{2}\int_0^{2\pi R}\int_0^L \Big[A_{11}\varepsilon_x^2 + 2A_{12}\varepsilon_x\varepsilon_y + 2A_{16}\varepsilon_x\gamma_{xy} + 2A_{26}\varepsilon_y\gamma_{xy} + A_{22}\varepsilon_y^2 + A_{66}\gamma_{xy}^2 + D_{11}\kappa_x^2$$

$$+ 2D_{12}\kappa_x\kappa_y + 2D_{16}\kappa_x\kappa_{xy} + 2D_{26}\kappa_y\kappa_{xy} + D_{22}\kappa_y^2 + D_{66}\kappa_{xy}^2 - N_0\left(\frac{\partial w}{\partial x} + \frac{\partial w_o}{\partial x}\right)^2\Big]\,\mathrm{d}x\mathrm{d}y. \tag{14}$$

Substitution of eqn (1) into the above formula leads to the energy expression in terms of displacements u, v, w:

$$\Pi = \frac{1}{2}\int_0^{2\pi R}\int_0^L \Bigg[A_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^2 + 2A_{12}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\left[\frac{\partial v}{\partial y} + \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$+ 2A_{16}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]$$

$$+ 2A_{26}\left[\frac{\partial v}{\partial y} + \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] + A_{22}\left[\frac{\partial v}{\partial y} + \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]^2$$

$$+ A_{66}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]^2 + D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

$$+ 4D_{16}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x\partial y} + 4D_{26}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x\partial y} + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2$$

$$+ 4D_{66}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - N_0\left(\frac{\partial w}{\partial x} + \frac{\partial w_0}{\partial x}\right)^2\Bigg]\,\mathrm{d}x\mathrm{d}y. \tag{15}$$

In Koiter's energy criterion of elastic stability, variations of energy are performed at the fundamental (pre-buckling) state.

The second variation of the energy for buckling modes is

$$P_2[\underline{u}] = \frac{1}{2}\int_0^{2\pi R}\int_0^L \Bigg[A_{11}\left(\frac{\partial u}{\partial x}\right)^2 + 2A_{12}\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial y} + \frac{w}{R}\right) + 2A_{16}\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$+ 2A_{26}\left(\frac{\partial v}{\partial y} + \frac{w}{R}\right)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + A_{22}\left(\frac{\partial v}{\partial y} + \frac{w}{R}\right)^2 + A_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2$$

$$+ D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 4D_{16}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x\partial y} + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2$$

$$+ 4D_{26}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x\partial y} + 4D_{66}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - N_0\left(\frac{\partial w}{\partial x}\right)^2\Bigg]\,\mathrm{d}x\mathrm{d}y. \tag{16}$$

We will discuss the effect of the most critical type of axisymmetric geometrical imperfection $w_0(x) = -\mu h_0 \cos(2px/R)$ (Koiter, 1963; Tennyson *et al.*, 1971) where h_0 is the nominal thickness of the shell, μ is the non-dimensional parameter giving the amplitude of the imperfection, and p is the wave number of the axisymmetric classical buckling mode, which is given by (Tennyson *et al.*, 1971):

$$p^2 = \frac{R}{4\sqrt{a_{22}^* d_{11}^*}}. \tag{17}$$

We supplement the second variation with the additional bilinear term due to geometric initial imperfection

$$P_{11}[\underline{u}_0, \underline{u}] = -N_0 \int_0^{2\pi R} \int_0^L \frac{\partial w}{\partial x} \frac{\mathrm{d}w_0}{\mathrm{d}x} \mathrm{d}x\mathrm{d}y. \tag{18}$$

The third variation of the energy reads

$$\begin{aligned} P_3[\underline{u}] = \frac{1}{2}\int_0^{2\pi R}\int_0^L \Bigg\{ & A_{11}\frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2 + A_{12}\left[\frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial u}{\partial y} + \frac{w}{R}\right)\right] \\ & + 2A_{16}\left[\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \\ & + 2A_{26}\left[\left(\frac{\partial v}{\partial y} + \frac{w}{R}\right)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \\ & + A_{22}\left(\frac{\partial v}{\partial y} + \frac{w}{R}\right)\left(\frac{\partial w}{\partial y}\right)^2 + 2A_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Bigg\}\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{19}$$

We now assume that the buckling modes of the shell with a uniform thickness remain a good approximation for the buckling modes of the shell with small thickness variations. We are at least ensured that the critical load obtained in this was is, by the energy principle, an upper bound for the actual critical buckling load.

According to the study of Tennyson *et al.* (1971), the following expression for the buckling mode can be adopted for the laminated cylindrical shell with the aforementioned axisymmetric initial imperfection w_0:

$$w = b\cos\frac{2px}{R} + C_n\cos\frac{px}{R}\cos\frac{ny}{R} \tag{20}$$

where b and C_n are constants, n is the number of waves in the circumferential direction. If we recall the shell equilibrium equations in terms of displacements u, v and w (Vinson Sierakowski, 1986)

$$\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{12} & L_{22} & L_{23} \\ L_{13} & L_{23} & L_{33} \end{pmatrix}\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ N_0\dfrac{\partial^2 w}{\partial x^2} \end{Bmatrix} \tag{21}$$

where operators L_{ij} are

$$L_{11} = a_{11}\frac{\partial^2}{\partial x^2} + 2a_{16}\frac{\partial^2}{\partial x\partial y} + a_{66}\frac{\partial^2}{\partial y^2}, \quad L_{12} = a_{16}\frac{\partial^2}{\partial x^2} + (a_{12}+a_{66})\frac{\partial^2}{\partial x\partial y} + a_{26}\frac{\partial^2}{\partial y^2}$$

$$L_{13} = \frac{1}{R}\left(a_{12}\frac{\partial}{\partial x} + a_{26}\frac{\partial}{\partial y}\right), \quad L_{22} = a_{66}\frac{\partial^2}{\partial x^2} + 2a_{26}\frac{\partial^2}{\partial x\partial y} + a_{22}\frac{\partial^2}{\partial y^2},$$

$$L_{23} = \frac{1}{R}\left(a_{26}\frac{\partial}{\partial x} + a_{22}\frac{\partial}{\partial y}\right)$$

$$L_{33} = \frac{a_{22}}{R^2} + d_{11}\frac{\partial^4}{\partial x^4} + 4d_{16}\frac{\partial^4}{\partial x^3 \partial y} + 2(d_{12}+2d_{66})\frac{\partial^4}{\partial x^2 \partial y^2} + 4d_{26}\frac{\partial^4}{\partial x \partial y^3} + d_{22}\frac{\partial^4}{\partial y^4} \quad (22)$$

we can obtain the expressions for u and v as follows

$$u = -\frac{a_{12}}{2pa_{11}}b\sin\frac{2px}{R} + Q_n C_n \sin\frac{px}{R}\cos\frac{ny}{R}$$

$$v = K_n C_n \cos\frac{px}{R}\sin\frac{ny}{R} \quad (23)$$

where

$$K_n = -\frac{n(p^2 a_{11}a_{22} + n^2 a_{66}a_{22} - a_{12}^2 p^2 - a_{66}a_{12}p^2)}{(a_{11}p^2 + a_{66}n^2)(a_{66}p^2 a_{22}n^2) - (a_{12}+a_{66})^2 p^2 n^2}$$

$$Q_n = -\frac{pa_{66}(p^2 a_{12} - n^2 a_{22})}{(a_{11}p^2 + a_{66}n^2)(a_{66}p^2 + a_{22}n^2) - (a_{12}+a_{66})^2 p^2 n^2}. \quad (24)$$

It should be mentioned that in deriving solution (23), we used an assumption in the studies of Hirano (1979) and Tasi (1966) that the coupling stiffnesses A_{16}, A_{26}, D_{16}, and D_{26} are zero. They are identically zero for cross-ply laminates. When the laminate is composed of many layers, these coupling stiffnesses are small and can be neglected.

In our previous numerical analysis of composite shells with axisymmetric thickness variations (Li *et al.*, 1994), we have shown that, in the absence of the geometric imperfection, the thickness variation with wave number being twice that of the classical buckling mode ($p_1 = 2p$) has the most degrading effect of the buckling load. This result is also observed for isotropic shells (Koiter *et al.*, 1994a, 1994b). Now we are interested in the combined effect of the most critical geometric imperfection and the most detrimental thickness variation on the load-carrying capacity.

Substituting eqn (20) and (23) into the second and third variations, we obtain, after retaining only the first order terms in ε:

$$P_2[\underline{u}] = \frac{C_n^2 \pi L}{4R^3}[d_{22}n^4 + 2d_{12}n^2p^2 + 4d_{66}n^2p^2 + d_{11}p^4 + a_{22}(1+K_n n)^2 R^2 - N_0 p^2 R^2 + 2a_{12}(1+K_n n)Q_n pR^2 + a_{11}Q_n^2 p^2 R^2 + a_{66}(K_n p + nQ_n)^2 R^2] + \frac{1}{2R^3}b^2\left[16d_{11}p^4\left(1-\frac{3}{2}\varepsilon\right) - \frac{a_{12}^3 R^2}{a_{11}}\left(1-\frac{1}{2}\varepsilon\right) + a_{22}R^2\left(1-\frac{1}{2}\varepsilon\right) - 4N_0 R^2 p^2\right] \quad (25)$$

$$P_{11}[\underline{u_0}, \underline{u}] = \frac{4bh_0 N_0 p^2 \mu\pi L}{R} \quad (26)$$

$$P_3[\underline{u}] = \frac{bC_n^2 \pi L}{8R^2}\left\{a_{12}\left[\left(-1+\frac{1}{2}\varepsilon\right)p^2 + 4\left(1+\frac{1}{2}\varepsilon\right)(1+K_n n)p^2\right] + a_{22}\left(1-\frac{1}{2}\varepsilon\right)n^2 + \frac{a_{12}^2 n^2}{a_{11}} \times\left(-1+\frac{1}{2}\varepsilon\right) + a_{12}\left(1-\frac{1}{2}\varepsilon\right)p^2 + 4a_{11}p^3 Q_n\left(1+\frac{1}{2}\varepsilon\right) - 4a_{66}(1+\tfrac{1}{2}\varepsilon)np(K_n p + Q_n n)\right\}. \quad (27)$$

The energy expression to be considered is

$$P_2[\underline{u}]+P_{11}[\underline{u}_0,\underline{u}]+P_3[\underline{u}]. \tag{28}$$

The equations for the initial post-buckling behavior are furnished by setting the partial derivatives of the energy expression (28) with respect to b and C_n equal to zero:

$$\begin{aligned}&\frac{b}{R^3}\left[16d_{11}p^4\left(1-\frac{3}{2}\varepsilon\right)-\frac{a_{12}^2R^2}{a_{11}}\left(1-\frac{1}{2}\varepsilon\right)+a_{22}R^2\left(1-\frac{1}{2}\varepsilon\right)-4N_0R^2p^2\right]\\&\quad+\frac{4N_0h_0p^2\mu}{R}+\frac{C_n^2}{8R^2}\left\{a_{12}\left[\left(-1+\frac{1}{2}\varepsilon\right)p^2+4\left(1+\frac{1}{2}\varepsilon\right)(1+K_nn)p^2\right]\right.\\&\quad+a_{22}\left(1-\frac{1}{2}\varepsilon\right)n^2+\frac{a_{12}^2n^2}{a_{11}}\left(-1+\frac{1}{2}\varepsilon\right)+a_{12}\left(1-\frac{1}{2}\varepsilon\right)p^2\\&\quad\left.+4a_{11}p^3Q_n\left(1+\frac{1}{2}\varepsilon\right)-4a_{66}\left(1+\frac{1}{2}\varepsilon\right)np(K_np+Q_nn)\right\}=0\end{aligned} \tag{29}$$

and

$$\begin{aligned}&\frac{C_n}{2R^3}[d_{22}n^4+2d_{12}n^2p^2+4d_{66}n^2p^2+d_{11}p^4+a_{22}(1+K_nn)^2R^2\\&\quad-N_0p^2R^2+2a_{12}(1+K_nn)Q_npR^2+a_{11}Q_n^2p^2R^2+a_{66}(K_np+nQ_n)^2R^2]\\&\quad+\frac{bC_n}{4R^2}\left\{a_{12}\left[\left(-1+\frac{1}{2}\varepsilon\right)p^2+4\left(1+\frac{1}{2}\varepsilon\right)(1+K_nn)p^2\right]+a_{22}\left(1-\frac{1}{2}\varepsilon\right)n^2\right.\\&\quad+\frac{a_{12}^2n^2}{a_{11}}\left(-1+\frac{1}{2}\varepsilon\right)+a_{12}\left(1-\frac{1}{2}\varepsilon\right)p^2+4a_{11}p^3Q_n\left(1+\frac{1}{2}\varepsilon\right)\\&\quad\left.-4a_{66}\left(1+\frac{1}{2}\varepsilon\right)np(K_np+Q_nn)\right\}=0.\end{aligned} \tag{30}$$

With the solution $C_n = 0$ from eqn (30), eqn (29) yields

$$b=-\frac{4N_0h_0p^2\mu R^2}{16d_{11}p^4\left(1-\frac{3}{2}\varepsilon\right)-\frac{a_{12}^2R^2}{a_{11}}\left(1-\frac{1}{2}\varepsilon\right)+a_{22}R^2\left(1-\frac{1}{2}\varepsilon\right)-4N_0R^2p^2}. \tag{31}$$

Bifurcation buckling with respect to the asymmetric mode with amplitude C_n occurs at

$$\begin{aligned}b=&-\frac{2}{R}[d_{22}n^4+2d_{12}n^2p^2+4d_{66}n^2p^2+d_{11}p^4+a_{22}(1+K_nn)^2R^2\\&-N_0p^2R^2+2a_{12}(1+K_nn)Q_npR^2+a_{11}Q_n^2p^2R^2+a_{66}(K_np+nQ_n)^2R^2]\\&\div\left\{a_{12}\left[\left(-1+\frac{1}{2}\varepsilon\right)p^2+4\left(1+\frac{1}{2}\varepsilon\right)(1+K_nn)p^2\right]+a_{22}\left(1-\frac{1}{2}\varepsilon\right)n^2\right.\\&+\frac{a_{12}^2n^2}{a_{11}}\left(-1+\frac{1}{2}\varepsilon\right)+a_{12}\left(1-\frac{1}{2}\varepsilon\right)p^2+4a_{11}p^3Q_n\left(1+\frac{1}{2}\varepsilon\right)\\&\left.-4a_{66}\left(1+\frac{1}{2}\varepsilon\right)np(K_np+Q_nn)\right\}.\end{aligned} \tag{32}$$

Equating the above two expressions for b, we obtain the equation for the critical buckling load N_0

$$\left[16d_{11}p^4\left(1-\frac{3}{2}\varepsilon\right)-\frac{a_{12}^2R^2}{a_{11}}\left(1-\frac{1}{2}\varepsilon\right)+a_{22}R^2\left(1-\frac{1}{2}\varepsilon\right)-4N_0R^2p^2\right]$$
$$\times[d_{22}n^4+2d_{12}n^2p^2+4d_{66}n^2p^2+d_{11}p^4+a_{22}(1+K_nn)^2R^2$$
$$-N_0p^2R^2+2a_{12}(1+K_nn)Q_npR^2+a_{11}Q_n^2p^2R^2+a_{66}(K_np+nQ_n)^2R^2]$$
$$-2N_0h_0p^2\mu R^3\left\{a_{12}\left[\left(-1+\frac{1}{2}\varepsilon\right)p^2+4\left(1+\frac{1}{2}\varepsilon\right)(1+K_nn)p^2\right]+a_{22}\left(1-\frac{1}{2}\varepsilon\right)n^2\right.$$
$$+\frac{a_{12}^2n^2}{a_{11}}\left(-1+\frac{1}{2}\varepsilon\right)+a_{12}\left(1-\frac{1}{2}\varepsilon\right)p^2+4a_{11}p^3Q_n\left(1+\frac{1}{2}\varepsilon\right)$$
$$\left.-4a_{66}\left(1+\frac{1}{2}\varepsilon\right)np(K_np+Q_nn)\right\}=0. \tag{33}$$

In solving eqn (33), integer search must be performed with respect to the circumferential wave number n to arrive at the lowest value of N_0.

We define the non-dimensional critical load parameter λ (sometimes referred to as *knockdown factor* in the literature) as

$$\lambda=\frac{N_0}{N_{cl}} \tag{34}$$

where N_{cl} is the classical buckling load in the absence of both initial imperfection and thickness variation. N_{cl} is given by (Vinson and Sierakowski, 1986)

$$N_{cl}=\min_{m,n}\{N_{m,n}\},\quad N_{m,n}=\left(\frac{L}{m\pi}\right)^2\frac{C_{11}C_{22}C_{33}+2C_{12}C_{23}C_{13}-C_{13}^2C_{22}-C_{23}^2C_{11}-C_{12}^2C_{33}}{C_{11}C_{22}-C_{12}^2} \tag{35}$$

where

$$C_{11}=A_{11}\left(\frac{m\pi}{L}\right)^2+A_{66}\left(\frac{n}{R}\right)^2,\quad C_{22}=A_{22}\left(\frac{n}{R}\right)^2+A_{66}\left(\frac{m\pi}{L}\right)^2$$
$$C_{33}=D_{11}\left(\frac{m\pi}{L}\right)^4+2(D_{12}+2D_{66})\left(\frac{m\pi}{L}\right)^2\left(\frac{n}{R}\right)^2+D_{22}\left(\frac{n}{R}\right)^4+\frac{A_{22}}{R^2}$$
$$C_{12}=(A_{12}+A_{66})\left(\frac{m\pi}{L}\right)\left(\frac{n}{R}\right),\quad C_{13}=\frac{A_{12}}{R}\left(\frac{m\pi}{L}\right),\quad C_{23}=\frac{A_{22}}{R}\left(\frac{n}{R}\right)+B_{22}\left(\frac{n}{R}\right)^3. \tag{36}$$

3. RESULTS AND DISCUSSION

Axial buckling loads can be determined from eqn (33) for composite cylindrical shells containing a small axisymmetric initial imperfection and a small axisymmetric thickness variation. For practical purposes, the results thus obtained should be considered conservative, since the most detrimental case of geometric imperfection and thickness variation is investigated. However, since we ignored in our derivation the higher order terms in ε, the results from the present study should not be deemed accurate for shells having large thickness variation. Also, the results may not be conservative from a design point of view because we are limited here to axisymmetric variations.

As a numerical example, we discuss shells made of carbon/epoxy laminae, whose elastic moduli are $E_1 = 13.75\times10^6$ psi, $E_2 = 1.03\times10^6$ psi, $\nu_{12} = 0.25$, $G_{12} = 0.42\times10^6$ psi. The shell is 6 inches in radius and 30 inches in length and is composed of ten equally thick layers, each being 0.012 inch thick. The laminate configuration is $[\theta/-\theta/\theta/-\theta/\theta]_{sym}$, with the fiber angle θ varying from 0° to 90°.

Solving eqn (33) numerically for the critical load N_0 with integer search performed simultaneously with respect to the circumferential buckling wave number n, and then non-dimensionalizing the result according to (34), we obtain the critical buckling load factor λ for different cases of thickness variation parameter ε and imperfection amplitude μ. The results are plotted in Figs 1 and 2. The results obtained here confirm numerically the previous first-order asymptotic formula

$$\lambda = 1-\varepsilon \tag{37}$$

which holds only for the axisymmetric buckling cases for composite shells without initial imperfection. It is interesting to note that as long as the axisymmetric buckling mode dominates, the buckling load reduction factor λ remains constant, irrespective of the change in the laminate construction. However, once the shell has an axisymmetric initial imperfection, the buckling mode becomes non-axisymmetric, and the buckling load reduction is strongly influenced by the stacking sequence of the laminae. Figure 3 depicts the results of the buckling load factor λ for shells of different laminate profiles, such as $[45°/-45°/45°/-45°/45°]_{sym}$ and $[16°/-16°/16°/-16°/16°]_{sym}$, together with the results for corresponding isotropic shells. It can be seen from this figure that the load-carrying capacity of composite shells is sensitive to thickness variation, and especially sensitive to initial geometric imperfection. The imperfection sensitivity is comparable to that of an isotropic shell. Although axisymmetric geometric initial imperfections cause most of the buckling load reduction, further degradation in the load-bearing capacity of the shell due to axisymmetric thickness variations should not be overlooked.

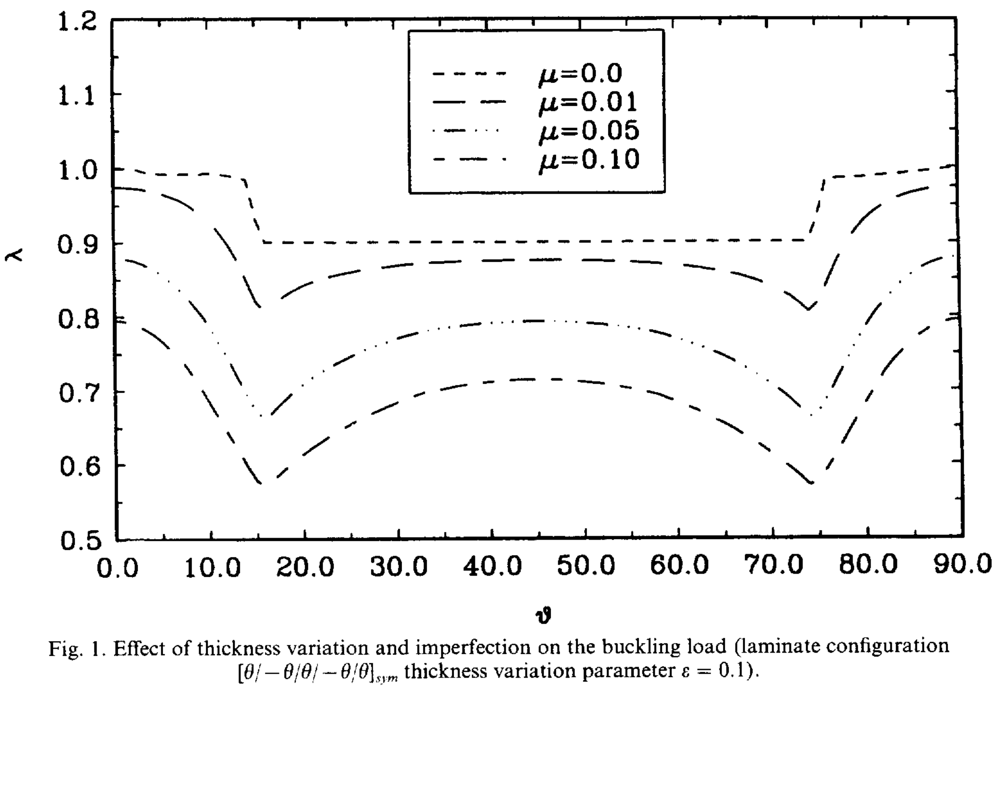

Fig. 1. Effect of thickness variation and imperfection on the buckling load (laminate configuration $[\theta/-\theta/\theta/-\theta/\theta]_{sym}$ thickness variation parameter $\varepsilon = 0.1$).

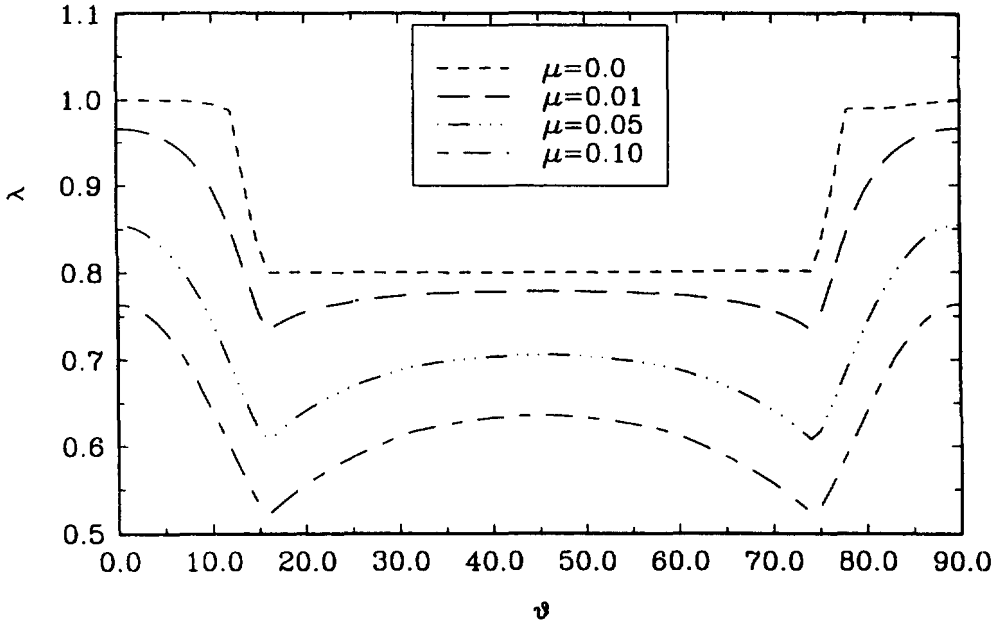

Fig. 2. Effect of thickness variation and imperfection on the buckling load ($[\theta/-\theta/\theta/-\theta/\theta]_{sym}$, $\varepsilon = 0.2$).

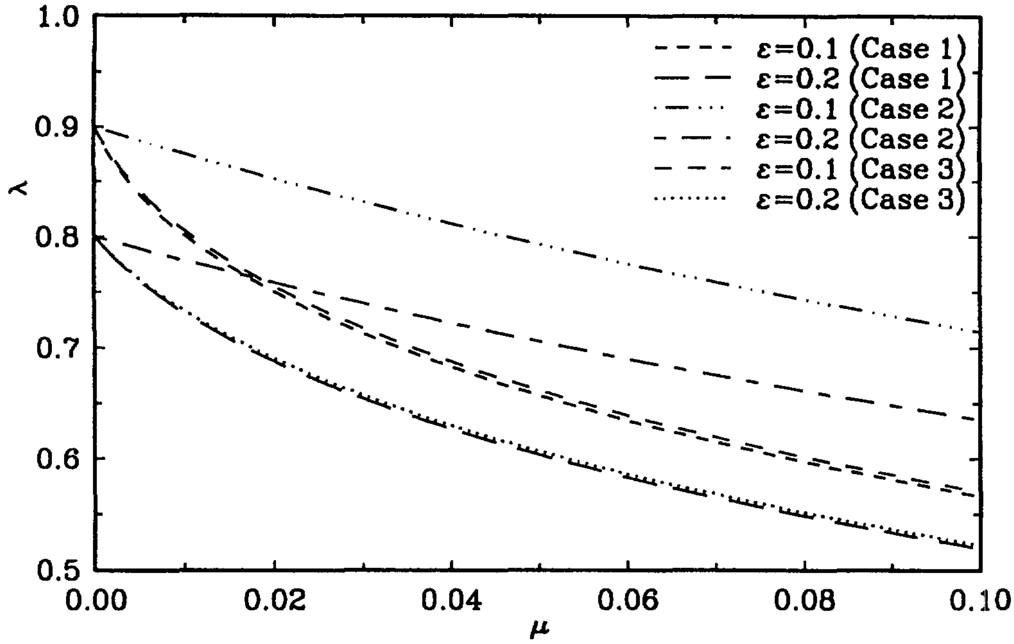

Fig. 3. Buckling load reductions in shells with different laminate configurations (Case 1: isotropic; Case 2: $[45/-45/45/-45/45]_{sym}$; Case 3: $[16/-16/16/-16/16]_{sym}$).

In order to check the accuracy of eqn (33), we used BOSOR4 (Bushnell, 1974), a computer code for stress, buckling and vibration of shells of revolution, to generate a set of comparable data for the non-dimensional critical load parameter λ. Since the classical buckling load has been used to non-dimensionalized the critical buckling load, it is necessary to check the results from eqn (35) with their counterparts from the numerical software so that a common basis can be established for the follow-up comparison of results for non-dimensional critical load λ. For this purpose, software package PANDA2, with use of either the shallow shell or Sanders' theories (Bushnell, 1987, 1996) was run in order to

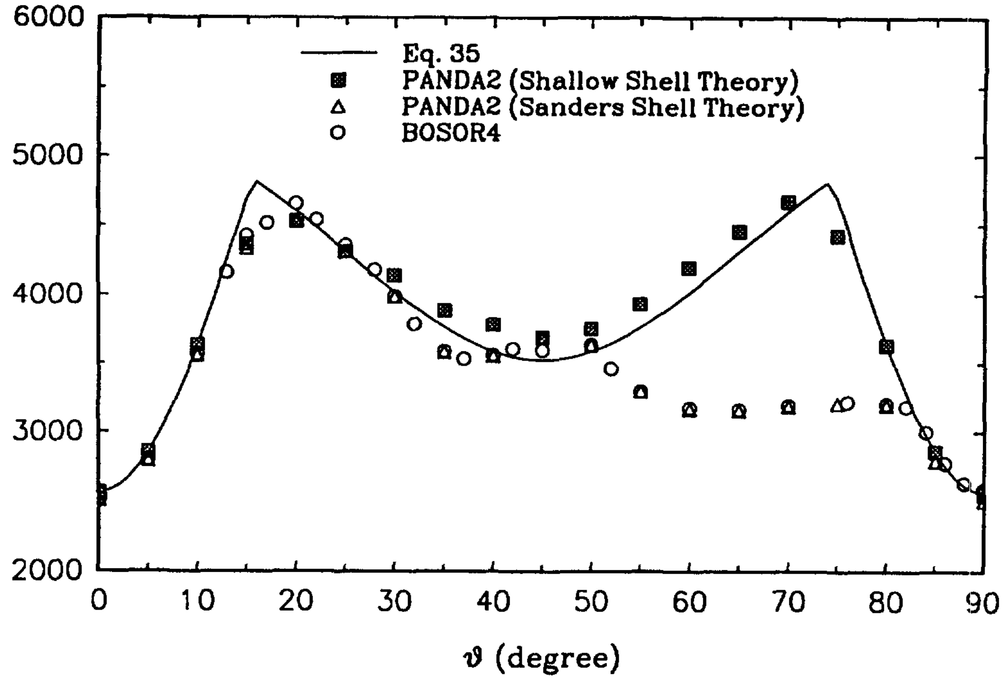

Fig. 4. Comparison of classical buckling load from different methods.

compute the classical buckling load N_{cl}. Predictions are plotted in Fig. 4, together with those from eqn (35) and those from BOSOR4, which is based on Sanders' equations.

Figure 4 shows that the classical buckling loads from different sources agree quite well except in the range $53° < \theta < 80°$. For this range eqn (35) and PANDA2 yield similar predictions with the shallow shell "switch" turned on in PANDA2. However, a significant discrepancy exists between predictions from shallow shell theory and Sander's theory. For $53° < \theta < 80°$ the shallow shell theory is significantly unconservative. A more refined theory is required for the accurate prediction of "classical" buckling load, N_{cl}, in this range of θ.

In the BOSOR4 models of the axisymmetrically imperfect shells, half of the 30 inch length of the cylindrical shell is represented, with symmetry conditions imposed at $x = 15$ inches. The 15 inch long BOSOR4 model is subdivided into six segments in order to get enough nodal points for a good convergence and in order to represent accurately the sinusoidal variation which is equal to the axial wavelength of the axisymmetric buckling mode of the perfect shell. The sixth segment, adjacent to the midlength plane of symmetry, is half as long. BOSOR4 can handle orthotropic walls with meridionally varying thickness. The shell wall in the BOSOR4 model has the same constitutive matrix as the 10-layer laminated shell with fiber angle $\theta = 16°$. (Note: in the BOSOR4 model the same off-diagonal "anisotropic" terms in the integrated constitutive law are assumed to be zero as is the case in the theory presented in this paper.)

Figure 5 displays the results of the non-dimensional critical load parameter λ obtained from the asymptotic formula (eqn 33) and from BOSOR4 for a 10-layer composite shell (laminate configuration: $[16°/-16°/16°/-16°/16°]_{sym}$) which contains both the initial imperfection and the thickness variation. It can be seen from this figure that the asymptotic formula predicts the knockdown factor quite accurately.

Finally, it is worth mentioning that as a special case if we let

$$a_{11} = a_{22} = \frac{Eh_0}{1-\nu^2}, \quad a_{12} = \nu a_{11}, \quad a_{66} = \frac{1-\nu}{2} a_{11}, \quad a_{16} = a_{26} = 0$$

$$d_{11} = d_{22} = \frac{Eh^3}{12(1-\nu^2)}, \quad d_{12} = \nu d_{11}, \quad d_{66} = \frac{1-\nu}{2} d_{11}, \quad d_{16} = d_{26} = 0 \tag{38}$$

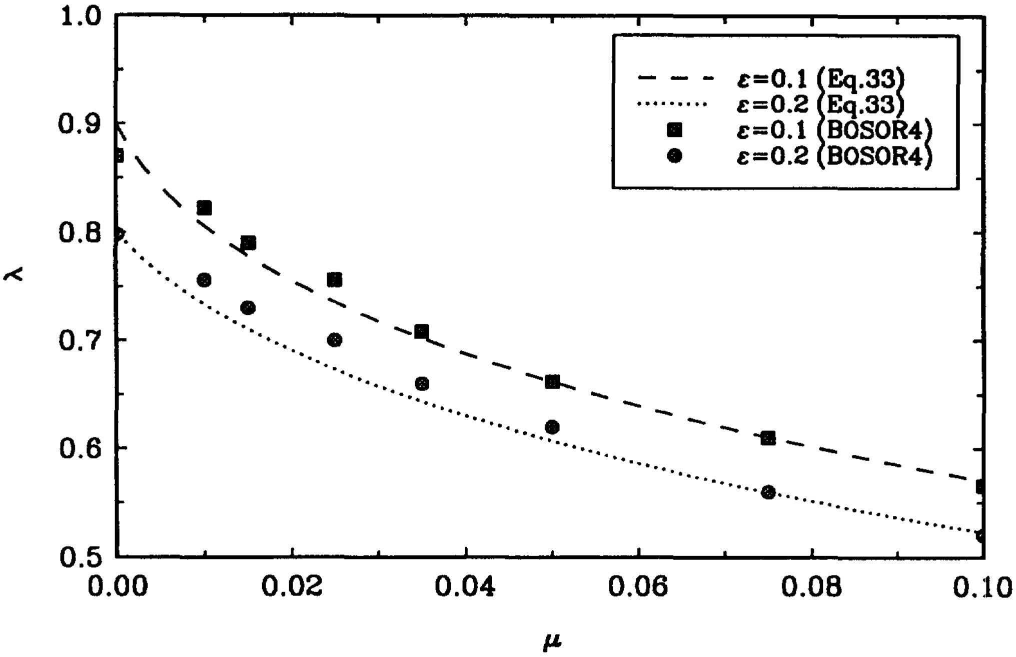

Fig. 5. Comparison of the non-dimensional critical load λ obtained from the asymptotic formula and BOSOR4.

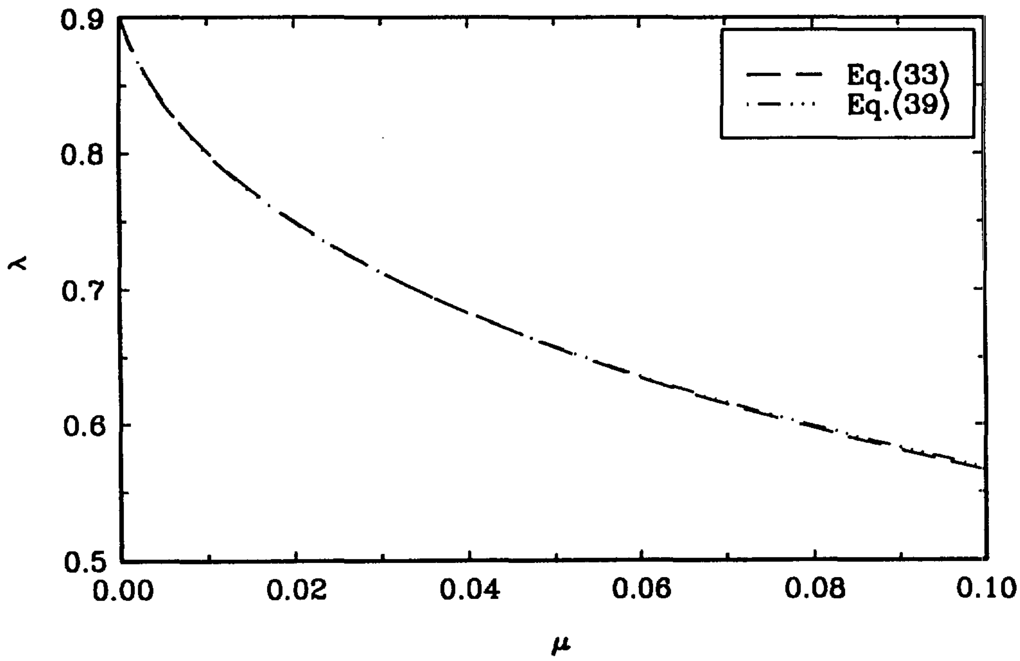

Fig. 6. Comparison of results using Koiter's circle and those using integer search for isotropic shells with thickness variation $\varepsilon = 0.2$.

where E is the Young's modulus, and v is the Poisson's ratio, and furthermore, if we select the asymmetric mode at the top of the Koiter's semi-circle (Koiter, 1980), that is, let $p = n = [\sqrt{3(1-v^2)}R^2/2h_0]^{1/2}$, eqn (33) reduces to its counterpart in the isotropic shell case,

$$(1-\lambda)(1-\varepsilon-\lambda)-\frac{3\sqrt{3(1-v^2)}}{2}\lambda\mu\left(1+\frac{1}{6}\varepsilon\right)=0. \tag{39}$$

Equation (39) is identical to eqn (21) in our previous work (Koiter *et al.*, 1994b) if the small term $\varepsilon/6$ is ignored compared with unity.

Figure 6 shows the comparison of results in the isotropic shell case using Koiter's semi-circle and those using integer search with respect to the circumferential wave number n. It is seen that the agreement is excellent.

One should stress here that in order to obtain good correlation of test and theory, the buckling load of perfect shells with nonlinear bending prebuckling effects should be calculated when the effects of boundary conditions become significant, as in the case for short shells and shells of intermediate length, for which the "boundary layer" length, $(rt)^{1/2}$ comprises a significant fraction of the entire length of the shell.

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